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Modeling Housing Price Dynamics and their Impact on the Cost of no–Negative–Equity–Guarantees for Equity Releasing Products

湖北省普通高等学校人文社会科学重点研究基地

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Modeling Housing Price Dynamics and their Impact on the Cost of no-Negative-Equity-Guarantees for Equity Releasing Products

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Abstract
We investigate model risk in pricing no-negative-equity guarantees (NNEGs) with the aim of identifying the housing risks involved in equity-release products. To analyze the regional and local effect in the house price modeling, we evaluate different models using the house price index (HPI) based on the cities of London, Manchester and Coventry and the UK nationwide HPI respectively. The ARMA-GARCH jump model that can capture the characteristics of jump persistence, autocorrelation and volatility clustering are proposed according to the model fittings. To investigate the model risk on the cost of NNEGs, we then derive the risk-neutral valuation framework using the conditional Esscher transform technique (Bühlmann et al. 1996). Our numerical analyses reveal that the housing model risk affects the costs of NNEGs significantly. In addition, the cost of NNEGs is significantly different for different cities due to localized effect. Therefore, the basis risk is large enough to matter when pricing NNEGs.

Keywords NNEGs · Equity-releasing products · House Price returns · Conditional Esscher transform

Introduction

The continuing global increase in life expectancy demands urgent consideration of the ways in which the retirement incomes of the elderly can be increased in order to ensure

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the maintenance of an acceptable standard of living. Although pension systems have long been the primary financial resource for elderly people, aging populations and increases in longevity on a global scale have put pension and annuity providers in untenable positions, such that the response by many providers has been unavoidable reductions in pension benefits (Antolin 2007). About 75% of the increasingly elderly populations around the world are now considered to have inadequate income upon their retirement; thus, governments are faced with the growing challenge of financing such aging populations. Clearly, therefore, development within the private markets of innovative financial products capable of increasing retirement income would be of significant benefit.

Many elderly people are considered to be "cash poor and equity rich" (McCarthy et al. 2002; Rowlingson 2006; Shan 2011). In the UK, for example, the aggregate non-mortgaged equity owned by people over the age of 65 years was found to be £1100 billion, whilst in the US, the median value of mortgage-free homes in the early part of the new century was found to be US$127,959, with more than 12.5 million elderly people having absolutely no mortgage debt (American Housing Survey for the United States 2005). Home equity therefore offers a potential alternative financial resource capable of meeting current shortfalls in retirement income; and indeed, equity-release products are designed exactly for this purpose, with homeowners receiving a lump sum and/or annuity in exchange for the transfer of some, or all, of the value of their house to a financial institution upon their death. The loan value is ultimately determined by the age of the borrower, the interest rate and the value of the property. Such equity-release products are available in several developed countries, including the US, the UK, France, Australia, Canada and Japan, with the major advantage for homeowners being that they can receive cash without having to leave the property. Due to the trend of population aging, a number of studies have estimated the potential demand for equity-release products. For example, across Europe as a whole, the report by Towers Watson (2014) estimate that there is potential for over €20bn to be released from equity release products each year and over €20bn 10 years.[1]

Equity-release products are widely offered by financial institutions, such as banks or insurance companies, but of course, there are risks involved for such institutions providing these products. The most obvious of these risks is the negative equity that such institutions may have to assume if the proceeds from the sale of the house prove to be less than the loan value paid out which is crossover risk. Equity-release mortgages differ from traditional mortgages, since the loans and accrued interest are required to be repaid when the borrower dies or leaves the house. Therefore management of these risks has become a crucial element for equity-release product providers in the continuing development of this market.

Writing of no-negative-equity guarantees (NNEGs) is the main method used to deal with the associated risks in equity-release products in the UK. NNEGs protect the borrowers by capping the redemption amount of the mortgage at the lesser amount of the face value of the loan or the sale proceeds of the property; thus, NNEGs can be viewed as a European put option on the mortgaged property. Since the effective

[1] This estimate projected in 2030 is based on the following conservative assumptions: the elderly population in Europe is 124 million; the overall home ownership in the population is 71%; average house price is €210,000; a 22% loan-to value and annual sales of 1/2%; no inflation.

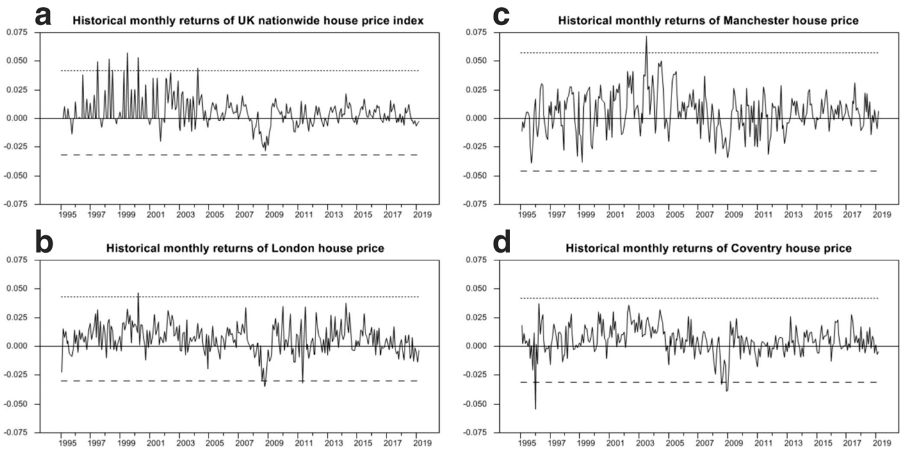

Fig. 1 Historical monthly returns of House Price Index (HPI), London, Manchester and Coventry, respectively

valuation of NNEGs has clearly become extremely important in developing an understanding of equity-release products, the primary aim of this study is to examine the housing model risk factors involved in the pricing of NNEGs.

In the continuing development of the pricing of equity-release products, the primary concern, thus far, has been shown to be house price risk (Kau et al. 1995), with the assumption in a number of the prior studies being that house prices are driven by a Geometric Brownian Motion (GBM) for reverse mortgages, which thereby facilitates the application of the Black and Scholes (1973) option pricing formula to NNEGs pricing.[2] Mortgage pricing models using the Black-Scholes approach have been introduced in several studies based upon the assumption that the house price process follows a standard stochastic process.[3] Further assuming that the house price index follows a GBM for derivative contracts based on the credit loss of mortgage portfolios, Duarte and McManus (2011) found that loss-based indices provided a better means of hedging credit risk in mortgage portfolios than indices based on house prices.

There are numerous similar examples to be found in the real estate literature; however, in many of the empirical investigations, two important properties have been found to be associated with house price return dynamics. Firstly, the log-return of house prices is found to be autocorrelated, and secondly, the volatility of the log-return of house prices is found to be time-varying or volatility clustering. Li et al. (2010) and Chen et al. (2010) therefore turned to the use of 'Autoregressive moving average - generalized autoregressive conditional heteroskedasticity' (ARMA-GARCH) models as their approach to capturing house price dynamics in the UK equity-release market and the US HECM program. However, there must also be consideration of the fact that house price return dynamics have been subject to abnormal shocks over recent years, the most obvious example of which is the 2008 subprime mortgage crisis.

[2] See, for example, Szymanoski (1994) and Wang et al. (2007).
[3] Examples include Ambrose and Buttimer (2000), Bardhan et al. (2006) and Liao et al. (2008).

The UK house price monthly returns from 1995 to 2019 are displayed in Fig. 1, with the details being obtained from the Nationwide House Price Index (HPI) and the city HPI based on London, Manchester and Coventry.[4] As we can see, housing price returns reveal significant jump risk when the monthly housing price returns is found to have changed by more than three standard deviations. The most significant downward jump occurred in 2008, following the outbreak of the subprime mortgage crisis. The pattern of return series is different between the city HPIs and the nationwide HPI. Given that the effects of such a downward jump are both systematic and non-diversifiable, this can lead to enormous problems within the general real estate market; thus, the jump effects in house prices have attracted considerable attention and related investigations over recent years.

Both Kau and Keenan (1996) and Chen et al. (2010) used the jump diffusion process to describe the changes in house prices, with the latter study demonstrating that abnormal shocks have significant impacts on mortgage insurance premiums. Chang et al. (2011) further extended the double exponential jump-diffusion model of Kou (2002) to consider the asymmetric jump risk in the pricing of mortgage insurance. On other hand, Eraker (2004), Duan et al. (2006, 2007), Maheu and McCurdy (2004) and Daal et al. (2007) find that accommodating for jumps effect in the log return and volatility considerably improves the model's fit for the return data of equity markets.

Nevertheless, despite the jump risk having been taken into consideration in the modeling of house price dynamics in numerous prior studies, it appears that each of these studies has failed to consider the important properties of volatility persistence and autocorrelation in the log returns and allows time-variation in jump component of the log returns and volatility.[5] We therefore aim to fill the gap within the extant literature by taking these factors into consideration. In specific terms, we study the jump dynamics in house price returns based upon an ARMA-GARCH specification which allows for both constant and dynamic jumps. Following Chan and Maheu (2002) and Maheu and McCurdy (2004),[6] we assume the distribution of jumps is to be Poisson with a time-varying conditional intensity parameter. In the empirical study, similar to the approach in Li et al. (2010), we focus on the UK equity-release market. However, in addition to the use of Nationwide HPI, we also consider the basis risk[7] and adopt the HPI in the cities of London, Manchester and Coventry to carry out our empirical

[4] There are localized effects in different cities and regions and HPI do not capture all of the semi-idiosyncratic risk that appears in housing. To consider basis risk, we also analyze some city/region indices in this study.

[5] Examples include Kau and Keenan (1996), Chen et al. (2010) and Chang et al. (2011).

[6] Chan and Maheu (2002), Eraker (2004), Maheu and McMcurdy (2004), Duan et al. (2006, 2007) and Daal et al. (2007) all consider the GARCH jump model for dealing with equity returns and find that accommodating for jumps effect in the log return and volatility considerably improves the model's fit for the return data of equity markets. Among them, Eraker (2004) and Duan et al. (2006, 2007), Chan and Maheu (2002), Maheu and McMcurdy (2004) and Daal et al. (2007) consider a dynamic jump setting. Duan et al. (2006) extended theory developed by Nelson (1990) and Duan (1997) by considering limiting models for the GARCH-jump process. In additional, Duan et al. (2007) provide empirical test of GARCH-jump model to price options, using data on S&P 500 index and the set of European options written on S&P 500 index. Further, Daal et al. (2007) proposed asymmetric GARCH-jump models that synthesize autoregressive jump intensities and volatility feedback in the jump component to fit for the dynamics of the equity returns in the US and emerging Asian stock markets. However, different to these literatures, we deal with house price return dynamics instead of equity returns. Thus, we further consider the ARMA-GARCH jump framework.

[7] The discrepancy between the returns on the mortgaged property.

analysis for the selection of the jump dynamic specifications based upon actual house price returns data. In order to facilitate our investigation of the jump effects in house price return modeling, we carry out a comparison between the fitting accuracy of the proposed ARMA-GARCH jump model and various other jump diffusion models, such as the Merton (1976) and Kou (2002) models, as well as the models proposed within the prior literature relating to NNEGs pricing, such as the GBM, ARMA-GARCH and ARMA-EGARCH models. Our empirical analyses, based upon three different data periods, reveal that the ARMA-GARCH jump model with dynamic jump specifications provides the best fit, according to both log-likelihood, Akaike information criteria (AIC) and Bayesian information criteria (BIC). The ARMA-GARCH dynamic jump model shows significant persistence in the conditional jump, which indicates that when designing equity-release products, we cannot ignore the jump risk associated with house price returns.

We provide findings are important to pricing of NNEGs and contributing to the extant literature on equity-release products in the following ways. Firstly, our study addresses the model risk in NNEGs pricing by comparing the costs based on various house price return models. Secondly, we derive our risk-neutral valuation framework with house price return dynamics based upon an ARMA-GARCH jump process using the conditional Esscher transform technique (Bühlmann et al. 1996). Thirdly, we also show the basis risk by identifying the cost of NNEGs is significant different based on the UK nationwide HPI and the city HPI in London, Manchester and Coventry due to there are localized effects in different cities and regions. Finally, our numerical findings reveal that ignoring the model risks will ultimately result in the pricing error of NNEGs.

The remainder of this paper is organized as follows. We construct an ARMA- GARCH jump model in Section 2 and then carry out empirical analyses to investigate the jump effect in house price returns. This is followed in Section 3 by the derivation of a risk-neutral valuation framework for NNEGs pricing under ARMA-GARCH jump models. A numerical investigation of the effects of housing models risk is subsequently carried out on NNEGs costs in Section 4. Finally, the conclusions drawn from this study are presented in Section 5.

Analysis of House Price Returns with Jumps for UK Equity Releasing Product

The Payoff of NNEGs with Equity Releasing Products

In the UK, an equity-release product must include the provision of a no- negative-equity-guarantees (NNEGs) in order to meet the Product Standards within the Statement of Principles of the Equity Release Council.[8] NNEGs protect the borrower by capping the redemption amount of the mortgage at the lesser of the face amount of the loan or the sale proceeds of the property; thus, the provision of NNEGs is similar in

[8] The Federal Home Loan Bank Board approved ERMs in 1979.

effect to the writing of a European put option on the mortgaged property. The effective valuation of NNEGs has clearly become an important issue. Because house price returns constitute the payoff of NNEGs, understanding the dynamic of house price returns is very critical in analyzing the cost of NNEGs.

Let's define the payoffs of the NNEGs for a 'roll-up' mortgage as an example.[9] We denote K_t as the outstanding balance of the loan and H_t represent the value of the mortgaged property. The amount repayable (outstanding balance) at time $T$is the sum of the principal, K, plus the interest accrued at a fixed roll-up rate; that is,

$$K_T = Ke^{\nu T}, \tag{1}$$

At the time that the loan becomes repayable, time T, if $H_t < K_t$, then the borrower pays H_t, and if $H_t > K_t$,then the borrower pays K_t. Once the loan is repaid, the provider receives an amount, K_t, plus the NNEGs payoff, which is:

$$-Max[K_t - H_t, 0], \tag{2}$$

or exactly the payoff of a short position on a European put option with strike price K_t written on an underlying mortgaged property, H_t. Nevertheless, the valuation of a NNEG is more complex than the valuation of a European equity put option, essentially because the house return model must be able to deal simultaneously with the preceding stylized facts such as autocorrelation, heteroskedasticity and jump effects. Neither the Black-Scholes nor the Merton jump option pricing formulae are appropriate for the valuation of NNEGs since the former assumes that the returns of the underlying asset follow a GBM, whilst the latter assumes that they follow a mixed-jump process. Thus, we validate the autocorrelated, heteroskedasticity and jump effects for house price returns first. In the following, we go on to analyze the specifications of an ARMA-GARCH jump model.

The ARMA(s,m)-GARCH(p,q) Jump Model for House Price Returns

Analysis of the properties of volatility clustering and autocorrelation effects with house price return dynamics has already been investigated by Chen et al. (2010) and Li et al. (2010). We extend their analysis by considering the jump effect with house price return dynamics based upon an empirical investigation (see Fig. 1); our analysis involves the construction of a house price return model capable of capturing the properties of volatility clustering and both jump and autocorrelation effects under Maheu and McCurdy (2004) framework.

We begin by investigating the house price returns data based upon time-series analysis, and then go on to develop the ARMA-GARCH jump model. Let $\left(\Omega; \Phi; P; (\Phi_t)_{t=0}^{T}\right)$ be a complete probability space, where P is the data-generating probability measure, with specifications for the conditional mean and conditional

[9] The most common types of payment options for equity-release products are lump sum (roll-up), terms, lines of credit, modified terms (combining lines of credit and term payments), tenure and modified tenure (combining lines of credit and tenure). Given that the roll-up mortgage has become the most popular payment option, our ongoing analysis focuses on this type of mortgage.

variance. Let H_t denote the UK house price and Y_t represent the house price return at time t. Y_t is defined as $\ln\left(\frac{H_t}{H_{t-1}}\right)$ and the proposed ARMA-GARCH jump model governing the return process is then expressed as:

$$Y_t = \ln\left(\frac{H_t}{H_{t-1}}\right) = \mu_t + \varepsilon_t, \tag{3}$$

The mean return follows an autoregressive moving average (ARMA) process as

$$u_t = c + \sum_{i=1}^{s} \vartheta_i Y_{t-i} + \sum_{j=1}^{m} \zeta_j \varepsilon,^{t-j} \tag{4}$$

where s is the order of the autocorrelation terms; m is the order of the moving average terms; ϑ_i is the i^{th}-order autocorrelation coefficient; ζ_j is the j^{th}-order moving average coefficient; ε_t is the total returns innovation observable at time t which is

$$\varepsilon_t = \varepsilon_{1,t} + \varepsilon_{2,t} \tag{5}$$

Extending from Maheu and McCurdy (2004),[10] we set two stochastic innovations in which the first component ($\varepsilon_{1,\,t}$)captures smoothly evolving changes in the conditional variance of returns and the second component ($\varepsilon_{2,\,t}$) causes infrequent large moves in returns and are denoted as jumps. $\varepsilon_{1,\,t}$ is set as a mean-zero innovation ($E[\varepsilon_{1,\,t}|\,\Phi_{t-1}] = 0$) with a normal stochastic forcing process as

$$\varepsilon_{1,t} = \sqrt{h_t} z_t,\, z_t \sim NID(0, 1), \tag{6}$$

And h_t denote the conditional variance of the innovations, given an information set of Φ_{t-1},

$$h_t = w + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j}, \tag{7}$$

where p is the order of the GARCH terms; q is the order of the ARCH term; α_i is the i^{th}-order ARCH coefficient; and β_j is the j^{th}-order GARCH coefficient. $\varepsilon_{1,\,t}$ is contemporaneously independent of $\varepsilon_{2,\,t}$. $\varepsilon_{2,\,t}$ is a jump innovation that is also conditionally mean zero ($E[\varepsilon_{2,\,t}|\,\Phi_{t-1}] = 0$) and we describe $\varepsilon_{2,\,t}$ in next subsection.

The Setting of Jump Dynamics

To capture the jump risk, the second component of innovation is employed to reflect the large change in price and modeled as

[10] Maheu and McCurdy (2004) consider the jump setting under a constant conditional mean of GARCH model. We deal with a jump ARMA-GARCH model and the likelihood function for parameter estimation is reconstructed.

$$\varepsilon_{2,t} = \sum_{k=1}^{Nt} V_{t,k} - \phi\lambda_t V_{t,k} \sim NID\left(\phi, \theta^2\right) \text{ for } k = 1, 2, \cdots \tag{8}$$

where $V_{t,\,k}$ denotes the jump size for the k^{th} jump with the jump size following the normal distribution with parameters, (ϕ, θ^2) and N_t is the jump frequency from time $t-1$ to t, distributed as a Poisson process with a time-varying conditional intensity parameter(λ_t); that is:

$$P(N_t = j|\Phi_{t-1}) = \frac{\exp(-\lambda_t)\lambda_t^j}{j!}, j = 0, 1, 2...., \tag{9}$$

where the parameter λ_t represents the mean and variance for the Poisson random variable, also referred to as the conditional jump intensity.

To facilitate our investigation of the jump effect on house price returns, we extend the work of Chan and Maheu (2002), Maheu and McCurdy (2004) and Daal et al. (2007) to specify λ_t as an ARMA form, which is

$$\lambda_t = \lambda_0 + \rho\lambda_{t-1} + \varsigma\psi_{t-1}, \tag{10}$$

where ρ measures jumps persistence. Since the ςvariable measures the sensitivity of the jump frequency (λ_t) to past shocks (ψ_{t-1}), with ψ_{t-1}representing the unpredictable component affecting our inference on the conditional mean of the counting process, N_{t-1}, then this suggests corresponding changes. We also investigate the constant jump effect, which represents a special case of Eq. (10) with the restriction of constant jump intensity($\lambda_t = \lambda_0$); this is imposed by setting $\rho = 0$ and $\varsigma = 0$.

The conditional jump intensity in this model is time-varying, with an unconditional value under certain circumstances. In order to derive the unconditional value of λ_t, we must first recognize that ψ_t is a martingale difference sequence with respect to Φ_{t-1}, because:

$$E[\psi_t|\Phi_{t-1}] = E[E[N_t|\Phi_t]|\Phi_{t-1}] - \lambda_t = \lambda_t - \lambda_t = 0, \tag{11}$$

Thus, $E[\psi_t = 0]$ and $Cov(\psi_t, \psi_{t-i}) = 0$, $i > 0$.

Another way of interpreting this result is to note that, by definition, ψ_t is nothing more than the rational forecasting error associated with updating the information set; that is, $\psi_t = E[N_t|\ \Phi_t] - E[N_t|\ \Phi_{t-1}]$. There are several important features in the conditional intensity model as noted by Maheu and McCurdy (2004). First, if the conditional jump intensity is stationary, ($|\rho| < 1$), then the unconditional jump intensity is equal to

$$E[\lambda_t] = \frac{\lambda_0}{1-\rho} \tag{12}$$

Second, to forecast λ_{t+i}, the multi-period forecasts of the expected number of future jumps are

$$E[\lambda_{t+i}|\Phi_{t-1}] = \begin{cases} \lambda_t i = 0 \\ \lambda_0\left(1+\rho+\cdots+\rho^{i-1}\right)+\rho^i\lambda_t i\geq 1 \end{cases} \tag{13}$$

Thus, the conditional jump intensity can be re-expressed as

$$\lambda_t = \lambda_0 + (\rho - \varsigma)\lambda_{t-1} + \varsigma E[N_{t-1}|\Phi_{t-1}] \tag{14}$$

Because the jump intensity residual is defined as

$$\psi_{t-1} = E[N_{t-1}|\Phi_{t-1}] - \lambda_{t-1} = \sum_{j=0}^{\infty} jP(N_{t-1} = j|\Phi_{t-1}) - \lambda_{t-1} \tag{15}$$

where $E[N_{t-1}|\Phi_{t-1}]$ is our ex post assessment of the expected number of jumps that occurred form $t-2$ to $t-1$, and $P(N_{t-1}=j|\Phi_{t-1})$ is called the filter and is the ex post inference on N_{t-1} give time $t-1$ information.

Note that, a sufficient condition to ensure $\lambda_t \geq 0$, for all $t > 1$, is $\lambda_0 > 0$, $\rho \geq \varsigma$, and $\varsigma > 0$. In addition, to forecast the conditional jump intensity, the startup value of λ_0 and $\psi_1$must be set. We follow Maheu and McCurdy (2004) to set $\lambda_0$as the unconditional value shown in Eq. (12), and $\psi_1 = 0$. More details regarding the ARMA jump intensity can be referred to Maheu and McCurdy (2004).

Parameter Estimation

The parameters of the ARMA-GARCH jump model can be estimated using the maximum likelihood estimation (MLE) method. The construction of the likelihood function is described as follows. Let $F_n(\Theta)$denote the log-likelihood function and $\Theta$is the parameter set governing the ARMA-GARCH jump model, which implies $\Theta = (c, \vartheta_s, \zeta_m, w, \alpha, \beta, \lambda_0, \rho, \varsigma, \phi, \theta)$ We aim to find the optimal parameters (Θ^*) to maximize the log-likelihood function. The log-likelihood function can be expressed as

$$F_n(\Theta) := \sum_{t=1}^{N} \log f(Y_t|\Phi_{t-1}, \Theta) \tag{16}$$

The conditional on j jumps occurring the conditional density of returns is Gaussian,

$$f(Y_t|N_t = j, \Phi_{t-1}, \Theta) = \frac{1}{\sqrt{2\pi\left(h_t + j\theta^2\right)}} \times \exp\left(-\frac{(Y_t - u_t + \phi\lambda_t - j\phi)^2}{2\left(h_t + j\theta^2\right)}\right). \tag{17}$$

In Eq. (16), the conditional density of return at time t ($f(Y_t|\Phi_{t-1}, \Theta)$) for calculating log-likelihood function can be obtained by integrating out the number of jumps as

$$\begin{aligned} f(Y_t|\Phi_{t-1}, \Theta) &= \sum_{j=0}^{\infty} f(Y_t|N_t = j, \Phi_{t-1}, \Theta) P(N_t = j|\Phi_{t-1}, \Theta) \\ &= \sum_{j=0}^{\infty} \frac{1}{\sqrt{2\pi\left(h_t + j\theta^2\right)}} \times \exp\left(-\frac{(Y_t - u_t + \phi\lambda_t - j\phi)^2}{2\left(h_t + j\theta^2\right)}\right) \cdot \frac{\exp(-\lambda_t)\lambda_t^j}{j!} \end{aligned} \tag{18}$$

Table 1 Summary statistics, 1995 m1–2019 m3

City / region indices	Variables	Mean	S.D.	Skewness[a,c]	Excess Kurtosis[a,c]	LB Q(34) Stats
UK HPI	Y_t	0.0049	0.0086	−0.3294**	1.7967***	629.5500***
	Y_t^2	0.0001	0.0002	3.0881***	12.0300***	348.2680***
London	Y_t	0.0063	0.0121	−0.0987	0.8455***	198.3601***
	Y_t^2	0.0002	0.0003	2.8567***	10.8469***	97.7572***
Manchester	Y_t	0.0056	0.0171	0.1448	0.4201	493.3531***
	Y_t^2	0.0003	0.0005	4.2522***	30.2662***	297.9754***
Coventry	Y_t	0.0053	0.0122	−0.6077***	2.4613***	346.392***
	Y_t^2	0.0002	0.0003	4.1012***	26.1271***	86.134***

[a] The skewness and excess kurtosis statistics include a test of the null hypotheses that each is zero (the population values if the series is i.i.d. Normal.)

[b] The LB Q (34) statistics refer to the null hypothesis of no serial correlation with 34 lags

[c] ** indicates significance at the 5% level; and *** indicates significance at the 1% level

where the conditional density of N_t ($P(N_t = j|\ \Phi_{t-1}, \Theta)$) is shown in Eq. (9). Since we assume the time-varying conditional intensity parameter (λ_t) follow an ARMA form as shown in Eq. (10), we need to work out the past shock(ψ_{t-1}) that affects the inference on the conditional mean of the counting process first. ψ_{t-1} is defined as

$$\psi_{t-1} = E[N_{t-1}|\Phi_{t-1}, \Theta]-\lambda_{t-1} = \sum_{j=0}^{\infty} jP(N_{t-1} = j|\Phi_{t-1}, \Theta)-\lambda_{t-1} \tag{19}$$

where $E[N_{t-1}|\ \Phi_{t-1}, \Theta]$ is given by Eq. (15). This expression could be estimated if $P(N_{t-1} = j|\ \Phi_{t-1}, \Theta)$ are known. Following Maheu and McCurdy (2004), the ex post probability of the occurrence of j jumps at time t-1 can be inferred using Bayes' formula as follows.

$$\begin{aligned} E[N_{t-1}|\Phi_{t-1}, \Theta] &= \sum_{j=0}^{\infty} jP(N_{t-1} = j|\Phi_{t-1}, \Theta) \\ &= \sum_{j=0}^{\infty} j \frac{f(Y_{t-1}|N_{t-1} = j, \Phi_{t-2}, \Theta)P(N_{t-1} = j|\Phi_{t-2}, \Theta)}{f(Y_{t-1}|\Phi_{t-2}, \Theta)} \\ &= \frac{\sum_{j=1}^{\infty} \frac{\exp(-\lambda_{t-1})\lambda_{t-1}^{j}}{j!} \frac{1}{\sqrt{2\pi\left(h_{t-1} + j\theta^2\right)}} \times \exp\left(-\frac{(Y_{t-1}-u_{t-1} + \phi\lambda_{t-1}-j\phi)^2}{2\left(h_{t-1} + j\theta^2\right)}\right)}{f(Y_{t-1}|\Phi_{t-2}, \Theta)} \end{aligned} \tag{20}$$

The details of Bayes' inference on calculating $E[N_{t-1}|\ \Phi_{t-1}, \Theta]$ is presented in Maheu and McCurdy (2004) Thus, by iterating on (10), (18) and (20), we can construct the log-likelihood function and obtain the maximum likelihood estimators. In addition, Eqs. (18), (19) and (20) involves an infinite summation depending on the jumps.[11] We

[11] Equation (18), (19) and (20) involve an infinite sum over the possible number of jumps, N_t. In practice, for our model estimated we found that the conditional Poisson distribution had zero probability in the tail for values of $N_t \geq 10$ and the likelihood and the parameter estimates converge.

Table 2 Parameter estimates and model fit of constant and dynamic jump models, 1995 m1–2019 m3

Parameters	UK HPI		London		Manchester		Coventry	
	ARMA(1,0)-GARCH(1,1) Models		ARMA(1,0)-GARCH(1,1) Models		ARMA(2,0)-GARCH(1,1) Models		ARMA(2,0)-GARCH(1,1) Models	
	Constant Jump	Dynamic Jump	Constant Jump	Dynamic Jump	Constant Jump	Dynamic Jump	Constant Jump	Dynamic Jump
Constant	0.0027***	0.0027***	0.0045***	0.0155***	−0.0004	−0.0005	0.0016	0.0016
	(0.0005)	(0.0005)	(0.0014)	(0.0053)	(0.0031)	(0.0002)	(0.0011)	(0.0001)
ϑ_1	0.4264***	0.4126***	0.2332***	0.6554***	0.3420***	0.3575***	0.3351***	0.3423***
	(0.0570)	(0.0599)	(0.0622)	(0.0158)	(0.0633)	(0.0740)	(0.0638)	(0.0686)
ϑ_2	–	–	–	–	0.1784***	0.2202***	0.2471***	0.2460***
	–	–	–	–	(0.0586)	(0.0750)	(0.0608)	(0.0670)
w	1.93e-07	6.74e-07	2.28e-05	9.25e-06	−1.26e-06	−8.77e-06	5.26e-06	5.83e-06
	(6.46e-07)	(7.60e-07)	(1.87e-05)	(2.41e-07)	(0.0586)	(3.85e-07)	(8.47e-06)	(1.02e-07)
α	0.0532**	0.0304*	0.0878*	0.0078*	0.0509	0.0452	0.0470	0.0480
	(0.0272)	(0.0441)	(0.0461)	(0.0054)	(0.0439)	(0.0293)	(0.0393)	(0.0374)
β	0.9142***	0.9296***	0.6224**	0.9730***	0.9246***	0.9190***	0.8389***	0.8944***
	(0.0402)	(0.0597)	(0.2111)	(0.0265)	(0.0519)	(0.0293)	(0.1527)	(0.1154)
λ_0	0.2514*	0.0192*	0.4346*	0.5647*	0.5247**	0.0881*	0.5888*	0.3514*
	(0.2189)	(0.0018)	(0.2734)	(0.1264)	(0.3580)	(0.0579)	(0.3710)	(0.0426)
ρ	–	0.9591***	–	0.8300***	–	0.8584***	–	0.7124***
	–	(0.0206)	–	(0.1089)	–	(0.1299)	–	(0.2189)
ς	–	0.3265	–	0.1826	–	0.0305	–	0.0143
	–	(0.6492)	–	(0.1512)	–	(0.0211)	–	(0.0101)
ϕ	0.0082**	0.0093*	0.0101**	0.0239**	0.0004	0.0177**	0.0271*	0.0129*
	(0.0033)	(0.0071)	(0.0049)	(0.0020)	(0.0007)	(0.0060)	(0.0065)	(0.0058)

Table 2 (continued)

Parameters	UK HPI		London		Manchester		Coventry	
	ARMA(1,0)-GARCH(1,1) Models		ARMA(1,0)-GARCH(1,1) Models		ARMA(2,0)-GARCH(1,1) Models		ARMA(2,0)-GARCH(1,1) Models	
	Constant Jump	Dynamic Jump	Constant Jump	Dynamic Jump	Constant Jump	Dynamic Jump	Constant Jump	Dynamic Jump
θ	0.0004	0.0005	0.0018*	0.0024*	0.0125*	0.0015*	0.0215*	0.0301*
	(0.0017)	(0.0084)	(0.0024)	(0.0014)	(0.0082)	(0.0018)	(0.0038)	(0.0044)
AIC	−7.0949	−7.1117	−6.0799	−6.1798	−5.6228	−5.6240	−6.3892	−6.4385
BIC	−6.9315	−6.9475	−5.9170	−5.9762	−5.4385	−5.4398	−6.2058	−6.2115
Log-likelihood	975.5491	982.8666	839.0351	852.8115	770.3333	777.5444	881.7035	890.7122

*, **, *** indicates significance at the 10%, 5% and 1% level, respectively. The standard errors of the estimation are shown in the parentheses

Table 3 Model selections, 1995 m1–2019m3

Nationwide/ City indices	Model	Log-Likelihood	AIC	BIC
UK HPI	Geometric Brownian Motion	854.9140	−5.8911	−5.7258
	ARMA-GARCH	958.4387	−6.9704	−6.8479
	ARMA-EGARCH	970.1280	−7.0812	−6.9188
	Merton jump	864.9137	−5.9304	−5.8672
	Double exponential jump diffusion	879.1243	−5.9941	−5.9109
	ARMA-GARCH Constant jump	975.5491	−7.0949	−6.9315
	ARMA-GARCH Dynamic jump	982.8666	−7.1117	−6.9475
London	Geometric Brownian Motion	757.3206	−5.8677	−5.4424
	ARMA-GARCH	830.9749	−6.0455	−5.7630
	ARMA-EGARCH	835.2531	−6.0525	−5.8303
	Merton jump	767.3208	−5.9470	−5.5838
	Double exponential jump diffusion	782.1498	−5.9814	−5.6610
	ARMA-GARCH Constant jump	839.0351	−6.0799	−5.9170
	ARMA-GARCH Dynamic jump	852.8115	−6.1798	−6.1798
Manchester	Geometric Brownian Motion	700.9627	−5.2894	−5.0941
	ARMA-GARCH	764.3475	−5.5791	−5.3357
	ARMA-EGARCH	770.1396	−5.6214	−5.3985
	Merton jump	710.1125	−5.3687	−5.1154
	Double exponential jump diffusion	718.9628	−5.4382	−5.1494
	ARMA-GARCH Constant jump	770.3333	−5.6228	−5.4385
	ARMA-GARCH Dynamic jump	777.5444	−5.6240	−5.4398
Coventry	Geometric Brownian Motion	800.7496	−6.1169	−5.9616
	ARMA-GARCH	867.8547	−6.3012	−6.1401
	ARMA-EGARCH	878.5910	−6.3130	−6.1587
	Merton jump	805.7497	−6.1362	−5.9729
	Double exponential jump diffusion	817.5987	−6.2895	−6.0914
	ARMA-GARCH Constant jump	881.7035	−6.3892	−6.2058
	ARMA-GARCH Dynamic jump	890.7122	−6.4385	−6.2115

find that truncation of the infinite sum in the likelihood at 10 captures all the tail probabilities and gleans sufficient precision in the estimation procedure.

Empirical Analysis of Model Fit

We examine the performance of the ARMA-GARCH jump model using time-series data and focus on an investigation into whether the conditional jump intensity is time-varying or constant. To consider the regional and local effect in the house price modeling, we evaluate the model using various HPI in the cities of London, Manchester and Coventry and the UK nationwide HPI. Our monthly data period runs from the 1995/1 to 2019/3, thereby providing a total of 291 monthly observations. As a robustness check, we also examine the results for different data periods (from the 2000/1 to 2019/3 and 2005/1 to 2019/3).

Table 4 Robustness check of model selections

City / region indices	Model	Log-Likelihood	AIC	BIC
Panel A: 2000 m1–2019 m3				
UK HPI	Geometric Brownian Motion	694.9567	−6.0127	−5.9928
	ARMA-GARCH	748.3367	−6.9682	−6.7983
	ARMA-EGARCH	758.2367	−7.0813	−6.8009
	Merton jump	704.9564	−6.0866	−6.0118
	Double exponential jump diffusion	708.1456	−6.1015	−6.0218
	ARMA-GARCH Constant jump	761.318	−7.0820	−6.8322
	ARMA-GARCH Dynamic jump	763.485	−7.1021	−6.9023
London	Geometric Brownian Motion	595.4259	−5.9428	−5.7829
	ARMA-GARCH	650.8875	−6.0730	−5.8576
	ARMA-EGARCH	656.8316	−6.0817	−5.8924
	Merton jump	605.4260	−5.9567	−5.8020
	Double exponential jump diffusion	610.8954	−5.9645	−5.8154
	ARMA-GARCH Constant jump	659.5754	−6.1071	−5.9081
	ARMA-GARCH Dynamic jump	672.3711	−6.2256	−5.9768
Manchester	Geometric Brownian Motion	565.9762	−5.2989	−5.2290
	ARMA-GARCH	605.4636	−5.6585	−5.4530
	ARMA-EGARCH	612.7325	−5.7264	−5.4560
	Merton jump	570.9784	−5.3129	−5.2381
	Double exponential jump diffusion	575.9847	−5.4087	−5.2541
	ARMA-GARCH Constant jump	613.5178	−5.7338	−5.4579
	ARMA-GARCH Dynamic jump	614.0964	−5.7392	−5.5135
Coventry	Geometric Brownian Motion	634.2999	−5.9914	−5.9001
	ARMA-GARCH	691.7557	−6.4350	−6.1601
	ARMA-EGARCH	695.3100	−6.4380	−6.1638
	Merton jump	642.3001	−5.9939	−5.9192
	Double exponential jump diffusion	650.1596	−6.0154	−5.9245
	ARMA-GARCH Constant jump	695.6294	−6.4410	−6.1672
	ARMA-GARCH Dynamic jump	698.0111	−6.4630	−6.2391
Panel B: 2005 m1-2019m3				
UK HPI	Geometric Brownian Motion	540.2675	−7.1202	−6.9591
	ARMA-GARCH	558.2675	−7.2794	−6.9841
	ARMA-EGARCH	565.8210	−7.3483	−7.1057
	Merton jump	548.6538	−7.1267	−6.9644
	Double exponential jump diffusion	550.1597	−7.1314	−6.9715
	ARMA-GARCH Constant jump	571.1226	−7.3693	−7.1090
	ARMA-GARCH Dynamic jump	571.3278	−7.3719	−7.1095
London	Geometric Brownian Motion	425.1710	−5.8897	−5.6288
	ARMA-GARCH	461.3396	−5.9714	−5.6478
	ARMA-EGARCH	467.1654	−5.9893	−5.6550
	Merton jump	435.1710	−5.8904	−5.6321
	Double exponential jump diffusion	438.4562	−5.8914	−5.6352

Table 4 (continued)

City / region indices	Model	Log-Likelihood	AIC	BIC
	ARMA-GARCH Constant jump	467.6271	−5.9952	−5.6715
	ARMA-GARCH Dynamic jump	470.3781	−6.0304	−5.7715
Manchester	Geometric Brownian Motion	400.4166	−5.5884	−5.4905
	ARMA-GARCH	449.8541	−5.8422	−5.6260
	ARMA-EGARCH	453.2809	−5.8867	−5.6289
	Merton jump	405.4166	−5.5931	−5.5009
	Double exponential jump diffusion	410.1432	−5.6032	−5.5107
	ARMA-GARCH Constant jump	453.7993	−5.8935	−5.6318
	ARMA-GARCH Dynamic jump	460.1478	−5.9547	−5.6611
Coventry	Geometric Brownian Motion	450.0314	−6.1204	−6.0535
	ARMA-GARCH	496.6803	−6.4503	−6.1941
	ARMA-EGARCH	505.5307	−6.4811	−6.1945
	Merton jump	461.0316	−6.1651	−6.0728
	Double exponential jump diffusion	478.1235	−6.2615	−6.1014
	ARMA-GARCH Constant jump	505.9544	−6.4865	−6.1952
	ARMA-GARCH Dynamic jump	525.1985	−6.5000	−6.2012

The summary statistics on the levels and squares of the log-return series for different HPI are reported in Table 1, from which it presents a clear evidence of time dependence using the modified Ljung-Box (LB) statistics (West and Cho 1995). These statistics, which are reported for autocorrelations of up to 34 lags, are found to be robust to validate the property of heteroskedasticity in house price returns. In addition, the modified LB statistics show strong serial correlation in both the levels and the squares of the house price return series in the UK nationwide HPI and in the cities of London, Manchester and Coventry respectively. The observations are consistent with Li et al. (2010) although they use UK nationwide HPI only in their empirical study.

We further investigate the jump dynamics for both dynamic and constant jump models under the framework of an ARMA(s,m)-GARCH(p,q) model. The parameters of these two ARMA-GARCH jump models are estimated by maximizing the conditional log-likelihood functions. The selection of the ARMA(s,m)-GARCH(p,q) model in this study is based upon the Box-Jenkins approach.[12] The details of the evaluation of our ARMA(s,m)-GARCH(p,q) jump models and the corresponding parameter estimates are presented in Table 2.

We evaluate the performance of the jump dynamics using log-likelihood, AIC and BIC.[13] The log-likelihood, AIC and BIC results indicate that the ARMA-GARCH dynamic jump model provides a better fit, with the persistence parameter (ρ) in this

[12] Although not reported here, the parameter estimates of the models are available upon request.
[13] AIC = −2/obs. ln(likelihood) + 2/ obs. × (No. of parameters) (Akaike 1973); BIC = −2/ obs. ln(likelihood) + ([No. of parameters] × ln[obs.]) / obs.; obs. is the sample size.
[0] The stochastic processes of these models are available upon request.

model being found to be statistically significant with range from 0.7124 to 0.9591 in the City of Coventry, Manchester, London and in UK HPI. This finding suggests that a high probability of many (few) jumps will also tend to be followed by a similarly high probability of many (few) jumps. Recall that ψ_t is the measurable shock constructed by econometricians using the ex post filter; thus, in a correctly-specified model, ψ_t should not display any systematic behavior.

In order to facilitate a thorough investigation in the present study of the importance of the jump effect in the modeling of house price returns, the existing models proposed in Chen et al. (2010) and Li et al. (2010) – which include the GBM, ARMA-GARCH and ARMA-EGARCH models – are also fitted to exactly the same series of housing returns. We further compare the performance of the ARMA-GARCH jump model with other jump diffusion models, such as the Merton (1976) and Kou (2002) models, both of which allow for jump effects, but do not consider the effects of autocorrelation and volatility persistence. We present the fitting results for each of the different models based on different house price return indices in Table 3.[14] The stochastic processes of these models are described in Appendix 1. The empirical results indicate the superiority of the ARMA-GARCH jump model over the existing house price return models, with the ARMA-GARCH dynamic jump model demonstrating further improvements on each of the other models based upon the log-likelihood, AIC and BIC values. The conclusion applies to both nationwide HPI and the HPI in different cities.

Although the jump effect is taken into consideration in the jump diffusion models, such as those proposed by Merton (1976) and Kou (2002), the performance of their models is nevertheless found to be inferior to that of the time-series models within which the effects of autocorrelation and volatility clustering are also taken into consideration; it therefore seems clear that a house price return model capable of simultaneously taking into consideration all three properties would represent an important contribution to this particular field of research.

As a check for the robustness of our results, we also investigate the model fit by considering different periods of the housing return data. The results for the 2000/1 to 2019/3 and 2005/1 to 2019/3 are shown in Table 4, For both sub-periods, the ARMA-GARCH dynamic jump model is still found to outperform each of the other models.

The results reported in Tables 3 to Table 4 confirm that the addition of jump dynamics improves the specification of the conditional distribution, as compared with the GBM, Merton jump, double exponential jump diffusion, ARMA-GARCH, ARMA- EGARCH and ARMA-GARCH constant jump models. In addition, the persistence parameter (ρ) governing the jump dynamic is statistically significant. It clearly indicates that jump risk in housing returns is significant and critical for pricing of NNEGs.[15]

[14] The stochastic processes of these models are available upon request.

[15] The persistence parameter (ρ) governing the jump model is estimated to be around 0.7124 to 0.9591, with statistical significance. We didn't report the entire parameter estimates here but they are available upon request.

Valuation of NNEGs

The Valuation Framework

Let $V(0, s)$ denote the no-arbitrage value of the NNEGs which is due at time s. The NNEGs becomes due when the borrower dies. Thus, for a person aged x at inception, the expected cost of the NNEGs, denoted as $V_{NNEG}(0, x)$, can be expressed as a series of European put options with different maturity dates. Under a discrete time steps, the fair value of the expected cost of a NNEGs is calculated as

$$V_{NNEG}(0,x) = \sum_{t=0}^{\omega-x-1} sp^{Q}(0,x)q_{s}^{Q}(0,x)V(0,s), \tag{21}$$

where ω is the maximal age of the borrower;${}_{s}p^{Q}(0,x)$ is the projected probability that a borrower aged x at inception will survive to age $x+s$ and $q_{s}^{Q}(0,x)$is the mortality that a borrower aged x at inception will die during the future time interval s to $s+1$ under the risk adjusted probability measure Q, or referring to as the risk-neutral measure.

The no-arbitrage value of $V(0, s)$ is calculated by discounting the payoff at time s under a risk-neutral measure Q, which is expressed as:

$$V(0,s) = E^{Q}[\exp^{-rs} Max[K_s - H_s, 0]]. \tag{22}$$

where r is the risk free interest rate.

To deal with the no-arbitrage value of $V(0, s)$ under the proposed ARMA-GARCH jump model, we need to obtain the risk-neutral process of the underlying housing price return. We use the conditional Esscher transform technique to derive the corresponding risk-neutral pricing. The corresponding process to obtain the risk-neutral valuation will be given in next subsection.

Risk-Neutral Valuation

To price NNEGs, we derive the corresponding risk-neutral return dynamic under the proposed ARMA-GARCH jump model by employing the conditional Esscher transform technique (Bühlmann et al. 1996). This technique has been widely used in the pricing of financial and insurance securities in an incomplete market since its introduction in 1932.[16] Siu et al. (2004) use the conditional Esscher transform for pricing derivatives when the underlying asset returns were found to follow GARCH processes. Recently, such technique has been extended to deal with pricing reverse mortgage products (Li et al. 2010; Chen et al. 2010; Yang 2011; Lee et al. 2012). To introduce the conditional Esscher transform technique, we define a sequence $\{\Lambda_t | t = j, j = 0, 1, \cdots, T\}$ be a Φ_t-adapted stochastic process:

[16] See, for example, Gerber and Shiu (1994), Bühlmann et al. (1996), Siu et al. (2004), Li et al. (2010) and Chen et al. (2010).

$$\Lambda_T = \prod_{t=1}^{T} \frac{\exp(aY_t)}{E[\exp(aY_t)|\Phi_{t-1}]} \tag{23}$$

where Y_t represents the house price return dynamic. The ARMA-GARCH jump model for capturing house price return under the real world measure can be referred to Eqs. (3)–(7). Bühlmann et al. (1996) has proved that $E(\Lambda_T) = 1$ and $E(\Lambda_T| \Phi_t) = \Lambda_t$. Equivalently, $\{\Lambda_t\}$is a martingale under P. We define a new martingale measure Q by

$$\frac{dQ}{dP}|\Phi_t = \Lambda_T \tag{24}$$

Then, under a risk neutral measure, Q, the housing price return dynamic then becomes

$$Y_t = \ln\left(\frac{H_t}{H_{t-1}}\right) = r - \frac{1}{2}h_t^* + \varepsilon_t^Q, \tag{25}$$

with $h_t^* = h_t + \left(\phi^2 + \theta^2\right)\lambda_t$ and $\varepsilon_t^Q = \varepsilon_t - a_t h_t^*$. ε_t^Q follows a normal distribution with mean 0 and variance h_t^* under measure Q. In other words, the house price return dynamic under measure Q is similar to the form under measure P, albeit with shifted parameters, that is. $Y_t|\Phi_{t-1} \sim N\left(r - \frac{1}{2}h_t^*, h_t^*\right)$. See Appendix 2 for the derivation of the risk-neutral ARMA-GARCH jump model in Eq. (25).

Mortality Dynamic: CBD Model

To modeling mortality dynamics in Eq. (21), as opposed to using the static mortality rate, we consider the longevity risk in NNEGs pricing and employ the CBD model (Cairns et al. 2006) to project future mortality rates. The CBD

Table 5 Base assumption of parameter values for the pricing of NNEGs

Parameters	Notation	Value
Risk-free interest rate (%)	r	1.878
Roll-up rate (%)	ν	2.000
Average delay in time (year)	δ	0.500
Market price of mortality risk	λ_m	0.175
Amount of loan advanced at inception	K	30,000
Initial property value for different ages, x, of borrowers (H_0)		
$x = 60$ Years		176,500
$x = 70$ Years		111,000
$x = 80$ Years		81,000
$x = 90$ Years		60,000

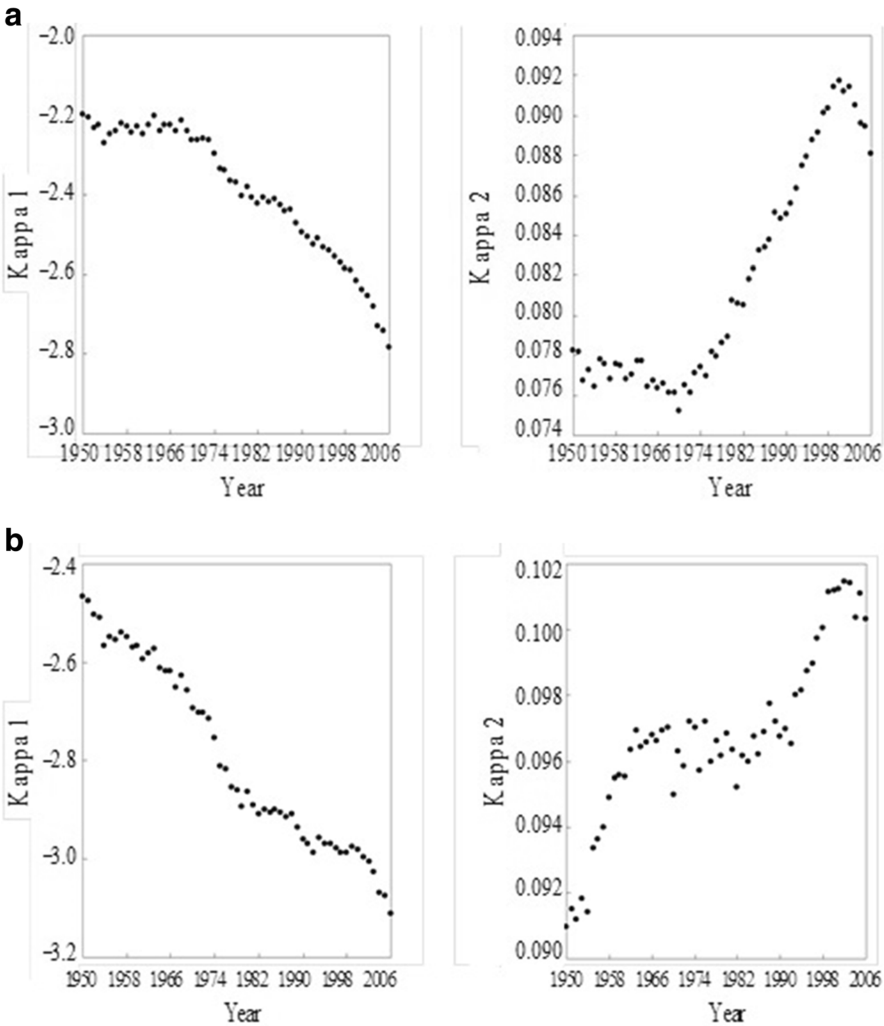

Fig. 2 **a** Estimated kappa values for male samples **b**. Estimated kappa values for female samples

model is attractive because it uses only a few parameters to obtain a good fit for the mortality probabilities of the elders; thus, this model has been widely adopted as a means of dealing with longevity risk for the elders (Wang et al. 2010; Yang 2011). Since the reverse mortgage products are issued for the elders, we also adopt the CBD model. Under the CBD model, the mortality rate for a person aged x dying before $x+1$ valued in year t, denoted as $q(t,x)$, is projected by:

$$\text{logit}\ q(t,x) = \kappa_t^{(1)} + \kappa_t^{(2)}\left(x - \bar{x}\right), \tag{26}$$

Table 6 The Cost of NNEGs under various house price return models for male Unit: %

Nationwide / city indices	Model	Age of Borrowers (x): years			
		$x=60$	$x=70$	$x=80$	$x=90$
UK HPI	Geometric Brownian Motion	3.845	3.035	2.126	1.376
		(0.023)	(0.021)	(0.017)	(0.007)
	ARMA-GARCH	5.545	4.305	2.966	1.816
		(0.026)	(0.023)	(0.019)	(0.007)
	ARMA-EGARCH	5.695	4.405	3.036	1.906
		(0.023)	(0.018)	(0.011)	(0.004)
	Merton jump	3.955	3.205	2.186	1.446
		(0.028)	(0.014)	(0.008)	(0.006)
	Double exponential jump diffusion	3.975	3.245	2.276	1.496
		(0.025)	(0.017)	(0.013)	(0.006)
	ARMA-GARCH Constant jump	5.985	4.655	3.186	1.946
		(0.032)	(0.029)	(0.024)	(0.012)
	ARMA-GARCH Dynamic jump	6.165	4.775	3.266	1.966
		(0.035)	(0.029)	(0.022)	(0.015)
London	Geometric Brownian Motion	7.835	5.322	3.012	1.757
		(0.019)	(0.016)	(0.012)	(0.008)
	ARMA-GARCH	9.535	6.592	3.852	2.197
		(0.025)	(0.022)	(0.017)	(0.004)
	ARMA-EGARCH	9.685	6.692	3.922	2.287
		(0.022)	(0.017)	(0.013)	(0.006)
	Merton jump	7.945	5.492	3.072	1.827
		(0.026)	(0.017)	(0.005)	(0.004)
	Double exponential jump diffusion	7.965	5.532	3.162	1.877
		(0.029)	(0.020)	(0.018)	(0.010)
	ARMA-GARCH Constant jump	9.975	6.942	4.072	2.327
		(0.034)	(0.026)	(0.019)	(0.011)
	ARMA-GARCH Dynamic jump	10.155	7.062	4.152	2.347
		(0.035)	(0.025)	(0.019)	(0.011)
Manchester	Geometric Brownian Motion	5.773	4.140	2.554	1.560
		(0.024)	(0.021)	(0.018)	(0.008)
	ARMA-GARCH	7.473	5.410	3.394	2.000
		(0.028)	(0.024)	(0.021)	(0.012)
	ARMA-EGARCH	7.623	5.510	3.464	2.090
		(0.031)	(0.024)	(0.020)	(0.009)
	Merton jump	5.883	4.310	2.614	1.630
		(0.026)	(0.022)	(0.011)	(0.007)
	Double exponential jump diffusion	5.903	4.350	2.704	1.680
		(0.026)	(0.023)	(0.015)	(0.012)
	ARMA-GARCH Constant jump	7.913	5.760	3.614	2.130
		(0.033)	(0.031)	(0.024)	(0.017)

Table 6 (continued)

Nationwide / city indices	Model	Age of Borrowers (x): years			
		$x = 60$	$x = 70$	$x = 80$	$x = 90$
	ARMA-GARCH Dynamic jump	8.093	5.880	3.694	2.150
		(0.035)	(0.031)	(0.024)	(0.020)
Coventry	Geometric Brownian Motion	5.027	3.713	2.389	1.489
		(0.026)	(0.019)	(0.014)	(0.010)
	ARMA-GARCH	6.727	4.983	3.229	1.929
		(0.030)	(0.025)	(0.018)	(0.014)
	ARMA-EGARCH	6.877	5.083	3.299	2.019
		(0.032)	(0.027)	(0.019)	(0.015)
	Merton jump	5.137	3.883	2.449	1.559
		(0.026)	(0.020)	(0.014)	(0.010)
	Double exponential jump diffusion	5.157	3.923	2.539	1.609
		(0.026)	(0.020)	(0.013)	(0.011)
	ARMA-GARCH Constant jump	7.167	5.333	3.449	2.059
		(0.034)	(0.030)	(0.024)	(0.017)
	ARMA-GARCH Dynamic jump	7.347	5.453	3.529	2.079
		(0.034)	(0.031)	(0.024)	(0.018)

Note: The standard error of the simulation is shown in the parentheses

where the parameter $\kappa_t^{(1)}$ represents the marginal effect of time on mortality rates; parameter $\kappa_t^{(2)}$ refers to the old age effect on mortality rates; and $\bar{x}$ is the mean age.[17] With the estimated values of $(\kappa_t^{(1)}, \kappa_t^{(2)})$, we can forecast the future mortality rates. In this study, we adopt Cairns et al. (2006)'s approach to estimate the parameters by using the least square method to fit the actual mortality curve and then project the $(\kappa_t^{(1)}, \kappa_t^{(2)})$ based upon a two-dimensional random walk with drift:

$$\kappa_{t+1} = \kappa_t + \mu + CZ_{t+1} \tag{27}$$

where $\kappa_t = \left[\kappa_t^{(1)}, \kappa_t^{(2)}\right]'$ and μ is a constant 2×1 vector; C is a constant 2×2 upper triangular matrix; and Z_t is a two-dimensional standard Gaussian process.

Equation (27) describes the dynamics of the random walk process κ_t under the real world probability measure, P, for projecting the mortality rate shown in Eq. (26). Let $p(t, x)$ denote the projected one-year survival rate in year t based upon the CBD model,

[17] We use the UK mortality data from 1950 to 2006 according to the human morality database (HMD) and the data ages cover from age 60 to 100. Therefore, the mean age is 80 in our model calibration.

Table 7 Pricing error under NNEGs costs of male for various models Unit: %

City / region indices	Model	Age of Borrowers (x years)			
		$x = 60$	$x = 70$	$x = 80$	$x = 90$
UK HPI	Geometric Brownian Motion	37.63	36.44	34.90	30.01
	ARMA-GARCH	10.06	9.84	9.18	7.63
	ARMA-EGARCH	7.62	7.75	7.04	3.05
	Merton jump	35.85	32.88	33.06	26.45
	Double exponential jump diffusion	35.53	32.04	30.31	23.91
	ARMA-GARCH Constant jump	2.92	2.51	2.45	1.02
	ARMA-GARCH Dynamic jump	–	–	–	–
London	Geometric Brownian Motion	22.85	24.64	27.46	25.14
	ARMA-GARCH	6.11	6.66	7.23	6.39
	ARMA-EGARCH	4.63	5.24	5.54	2.56
	Merton jump	21.76	22.23	26.01	22.16
	Double exponential jump diffusion	21.57	21.67	23.85	20.03
	ARMA-GARCH Constant jump	1.77	1.70	1.93	0.85
	ARMA-GARCH Dynamic jump	–	–	–	–
Manchester	Geometric Brownian Motion	28.67	29.59	30.86	27.44
	ARMA-GARCH	7.66	7.99	8.12	6.98
	ARMA-EGARCH	5.81	6.29	6.23	2.79
	Merton jump	27.31	26.70	29.24	24.19
	Double exponential jump diffusion	27.06	26.02	26.80	21.86
	ARMA-GARCH Constant jump	2.22	2.04	2.17	0.93
	ARMA-GARCH Dynamic jump	–	–	–	–
Coventry	Geometric Brownian Motion	31.58	31.91	32.31	28.38
	ARMA-GARCH	8.44	8.62	8.50	7.22
	ARMA-EGARCH	6.40	6.79	6.52	2.89
	Merton jump	30.08	28.79	30.61	25.01
	Double exponential jump diffusion	29.81	28.06	28.06	22.61
	ARMA-GARCH Constant jump	2.45	2.20	2.27	0.96
	ARMA-GARCH Dynamic jump	–	–	–	–

the projected probability in year t that a borrower aged x will survive to age $x + s$ is calculated by

$$p(t,x) = p(t,x)p(t+1,x+1)\cdots p(t+s,x+s-1). \tag{28}$$

To project the mortality rate under a risk-neutral probability measure Q, following Cairns et al. (2006), the dynamics become

$$\kappa_{t+1} = \kappa_t + \mu + C\left(\widetilde{Z}_{t+1} - \lambda_m\right) = \kappa_t + \widetilde{\mu} + C\left(\widetilde{Z}_{t+1}\right) \tag{29}$$

where $\widetilde{\mu} = \mu - C\lambda_m$.

$\widetilde{Z}_{t+1}$ in Eq. (29) is a standard two-dimensional normal random variable under Q. The vector $\lambda = (\lambda_{m1}, \lambda_{m2})$ represents the market price of the longevity risk associated with the respective processes of $Z_{1,\ t}$ and $Z_{2,\ t}$. where λ_{m1} is associated with level shift in mortality and λ_{m2} is associated with a tilt in morality. As in Cairns et al. (2006), we assume that the market price of risk λ_m is not updated over time; however, since there is no liquid market for systematic longevity risk, it is difficult to calibrate the risk-neutral survival probabilities using empirical data. Therefore, we follow the approach of Cairns et al. (2006) to carry out the calibrations as the parameter value of $\lambda_m = [0.175, 0.175]'$ for the pricing of NNEGs in the present study.

Numerical Analysis of the Costs of no-Negative-Equity Guarantees

Example Setting and Assumptions

In this section, we study the impacts of model risk and basis risk on the costs of NNEGs. We use Monte Carlo simulations to calculate the no-arbitrage value of NNEGs. Thus, we first generate 100,000 sample paths of the risk-neutral house price returns and then calculate the value of NNEGs ($V_{NNEG}(0, x)$)based on Eqs. (21) and (22). In addition, to implementing simulations, we assume that all deaths occur at midyear, and that δ is the average delay in the actual sale of the property in calculating the NNEGs.

For a comparison purpose, we follow Li et al. (2010) to set up the relevant assumptions for the NNEGs and list the information in Table 5. In addition, the parameter estimates for the housing price return for the ARMA-GARCH jump model can refer to Table 2 (in Section 2) and the parameter estimates for CBD mortality model as shown in Eq. (26) are plotted in Fig. 2.

House Price Risk Effects: Model Risk and Basis Risk

In the numerical analysis, the model risk is analyzed by calculating the cost of NNEGs under different models and the basis risk is examined by comparing the cost of NNEGs with the nationwide HPI and the local HPI based on different cities. We first discuss the model risk. Tables 6 and 7 show the cost of NNEGs under various house price return models including the Black-Scholes, Merton jump diffusion, double exponential jump diffusion, ARMA-GARCH and ARMA-EGARCH and the proposed ARMA-GARCH jump models and under different nationwide and local HPI respectively.

In Table 6, we express the cost of NNEGs as a percentage of the total amount of cash advanced. It shows that the cost of NNEGs is significantly different across different house price return models. For example, it raises from 3.845% to 6.165% for a male borrower aged 60 based on UK HPI; 7.835% to 10.155% on London HPI; 5.773% to 8.093% on Manchester HPI; 5.027% to 7.347% to Coventry HPI. The effect applies to the borrower at different ages and genders. However, the elder borrowers lead to a reduction in NNEGs costs because the coverage period of NNEGs is shorter.

We also measure the corresponding pricing error by comparing the cost of NNEGs with those calculated under the ARMA-GARCH dynamic jump model in Table 7. We find that the GBM model gives the lowest value for NNEGs and the pricing errors are far larger than those based on the ARMA-GARCH Dynamic jump model if we ignore the important properties of autocorrelation, volatility clustering and jump effects in house price dynamics. Our empirical analysis in Section 2 has already demonstrated that jump risk cannot be ignored when modeling house price dynamics, and indeed, taking the jump effect into account increases the overall cost of NNEGs. For example, for a male aged 60 based on the UK HPI, it results in the pricing error by approximate 10.06% based on ARMA-GARCH model. On the other hand, ignoring the dynamic jump function gives rise to a 2.92% pricing error. However, under different jump components, there is not much difference between Merton jump and Double exponential jump diffusion model. Furthermore, when the effects of autocorrelation and volatility clustering are considered, the pricing error decrease from 35.53% to 35.85%.

We further compare the cost of NNEGs using nationwide HPI with the city HPI. As shown in Table 6 to Table 7, the cost of NNEGs is significantly different in UK nationwide HPI and in the city HPI in London, Manchester and Coventry due to there are localized effects in different cities and regions. Thus, those differences are large enough to matter when pricing NNEGs. Li et al. (2010) actually use the UK nationwide HPI in their analysis. Although they didn't tackle the basis risk, they point out basis risk exists when using nationwide HPI. The basis risk is difficult to measure in the absence of individual house prices data. However, if we use the city HPI, it can help reduce the basis risk comparing with the cost calculated using nationwide HPI. In addition, the empirical results are consistent with Shao et al. (2015). Shao et al. (2015) has shown that pricing reverse mortgage loan based on an average house price index results in a substantial misestimation of the pricing in reverse mortgages. Therefore, the property's characteristics should be used in the pricing of reverse mortgages loans especially the location, the number of bathrooms and the land area.

Conclusions

In conjunction with the rapid growth in the equity-release market, there is growing demand for the development of effective risk management tools for these products. In the UK, equity-release products are commonly sold with no-negative-equity guarantee protection which caps the redemption amount at the lesser of the face amount of the loan or the sale proceeds. It therefore seems crucial for providers to have a firm understanding of the pricing of NNEGs.

Historical house price returns within the UK real estate market have experienced significant abnormal shocks, such as the subprime mortgage crisis in 2008, and since the providers of equity-release products assume substantial financial burdens when issuing NNEGs, it is extremely important for such providers to take into account the jump effects in house price returns when pricing these products. Despite this obvious requirement, this issue has not yet been dealt with in the prior literature; thus it is examined in the present study using an ARMA-GARCH jump model.

We contribute to the extant literature on NNEGs pricing in several ways. Firstly, having identified the jump risk as an intrinsic element of house price returns within the

UK mortgage market, we go on to propose the use of an ARMA-GARCH jump model. Secondly, our estimation of this model reveals that it offers a better fit than the various other house price return models proposed within the prior literature. Thirdly, we derive a risk-neutral framework for NNEGs pricing. Fourth, we examine the basis risk. We find the cost of NNEGs is significantly different between UK HPI and the HPI in the cities of London, Manchester and Coventry due to there are localized effects in different cities and regions. Finally, we show that if we ignore any housing dynamic properties, it will lead to large pricing error of NNEGs price. Since equity-release products are becoming increasingly important in globally aging societies, financial institutions issuing such products need to understand the impact of the model risk on NNEGs costs. We argue that the findings of our study can help such providers to manage the inherent risks.

In the light of our analysis, we suggest two areas for further research. Interest-rate risk is another important risk factor in analyzing the cost of NNEGs, since interest rates are a fundamental economic variable within any economy. Incorporation of the feature of stochastic interest rates in the valuation of contingent claims has been proposed in numerous studies within the extant financial literature.[18] It was also pointed out by Ho et al. (1997) that interest rate risk has become an increasingly important factor as a result of the term structure of interest rates affecting the value of options with long-term maturity. Kijima and Wong (2007) consider the pricing of equity-indexed annuities with stochastic interest rates, noting their substantial effects on the valuing of insurance policies with long horizons. A NNEG is similar to writing a long-duration European put option on the mortgaged property. Incorporating a stochastic interest rate in valuation of NNEGs should be a focus of a further study. In addition, this research illustrates the valuation of a NNEG with a simply policy assumption. However, other realistic features of a NNEG policy such as the prepayment design are well worth extending our valuation framework to examine the effect of policy features on the cost of a NNEG.

Appendix 1

This appendix provides a brief introduction to the house price return models investigated in the present study, with H_t denoting the house price at time t

The Geometric Brownian Motion Model

The random behavior of house prices is described under the Geometric Brownian Motion (GBM) model as:

$$\frac{dH_t}{H_t} = \mu dt + \sigma dW_t, \tag{A.1}$$

where μ is the drift term, σ refers to the volatility of house prices and W_t is a standard Brownian motion.

[18] Examples include Merton (1973), Rabinovitch (1989), Turnbull and Milne (1991) and Amin and Jarrow (1992).

The Merton Jump Diffusion Model

Considering the jump effect in a traditional GBM model, Merton (1976) proposed a jump diffusion model aimed at capturing the leptokurtic feature of asset pricing. The Merton model is expressed as:

$$\frac{dH_t}{H_t} = \mu dt + \sigma dW_t + dJ_t$$
$$J_T = \sum_{j=1}^{N_T} \left(V_j - 1\right) \tag{A.2}$$

where N_T follows a homogeneous Poisson process with λ; and V_j represents the jump size, which is an i.i.d. log-normal random variable with parameters, ϕ, θ^2; that is, $V_j \sim N(\phi, \theta^2)$. It should be noted that V_j is independent of both the Brownian motion, W, and the basic Poisson process, N.

The Double Exponential Jump Diffusion Model

The dynamics of the house price returns in the double exponential jump diffusion model are given by:

$$\frac{dH_t}{H_t} = \mu dt + \sigma dW_t + dJ_t J_T = \sum_{j=1}^{N_T} \left(V_j - 1\right), \tag{A.3}$$

where V_j is a sequence of i.i.d. non-negative random variables, such that $Y = \log(V_j)$ has an asymmetric double exponential distribution with density:

$$f_\gamma(y) = p \cdot \eta_1 e^{-\eta_1 y} 1 +^{\{y \geq 0} q \cdot \eta_2 e^{\eta_2 y} 1,^{\{y < 0} \eta_1 > 1, \eta_2 > 0,$$

where p, $q \geq 0$, $p + q = 1$. The $\eta_1 > 1$ condition is imposed in order to ensure that the house price, H_t, has a finite expectation. The respective means of the two exponential distributions are $1/\eta_1$ and $1/\eta_2$, and we assume that all sources of randomness within the model (N_t, W_t and γ_s) are independent.

The ARMA-GARCH Model

The ARMA(s,m)-GARCH(p,q) model is capable of capturing the properties of autocorrelation and volatility clustering, with two specifications (the conditional mean and conditional variance) being required for the development of this model; that is,

$$Y_t = c + \sum_{i=1}^{s} \vartheta_i Y +^{t-i} \sum_{j=1}^{m} \zeta_j \varepsilon +^{t-j} \varepsilon_t h_t = w + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j}, \tag{A.4}$$

where s is the order of the autocorrelation terms; m represents the order of the moving average terms; ϑ_i refers to the i^{th}-order autocorrelation coefficient; ζ_j is the j^{th}-order moving average coefficient; and ε_t is the i^{th} Gaussian innovation.

Furthermore, given an information set of $\Phi_{t\text{–}1}$, then h_t denotes the conditional variance of the innovations. We also let p be the order of the GARCH terms; q be the order of the ARCH term; α_i be the i^{th}-order ARCH coefficient; and β_j be the j^{th}-order GARCH coefficient.

The ARMA-EGARCH Model

As opposed to simply adopting a pure GARCH model, Nelson (1991) proposed an exponential GARCH (EGARCH) model in an attempt to allow for the asymmetric effects between positive and negative house returns. The ARMA(s,m)-EGARCH(p,q) model can be expressed as follows:

$$Y_t = c + \sum_{i=1}^{s} \vartheta_i Y +^{t-i} \sum_{j=1}^{m} \zeta_j \varepsilon +^{t-j} \varepsilon_t \ln(h_t) = w + \sum_{i=1}^{q} \alpha_i \widetilde{\varepsilon}_{t-i} + \sum_{i=1}^{q} \iota_i \left[|\widetilde{\varepsilon}_{t-i}| - E\left(|\widetilde{\varepsilon}_{t-i}| \right) \right] + \sum_{j=1}^{p} \beta_j \ln\left(h_{t-j}\right), \tag{A.5}$$

where $\widetilde{\varepsilon}_t = \varepsilon_t / \sqrt{h_t}$ is the standardized residual at time t.

An EGARCH process provides for the leverage effect using the leverage parameters, ι_i, which allows the conditional variance to respond to the asymmetric effect between the positive and negative innovations.

Appendix 2

We consider the housing price process follows an ARMA-GARCH jump model. Specifically, in a filtered probability space $\left(\Omega; \Phi; P; (\Phi_t)_{t=j}^{T} \right)$ be a complete probability space, the house price return process shown in Eq. (3) and (7) is given by

$$Y_t = \ln\left(\frac{H_t}{H_{t-1}} \right) = \mu_t + \varepsilon_t, h_t = w + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j}, \tag{B.1}$$

where $u_t = c + \sum_{i=1}^{s} \vartheta_i Y_{t-i} + \sum_{j=1}^{m} \zeta_j \varepsilon^{t-j}$ is the conditional mean function, given the time $t-1$ information Φ_{t-1}; s is the order of the autocorrelation terms; m is the order of the moving average terms; ϑ_i is the i^{th}-order autocorrelation coefficient; ζ_j is the j^{th}-order moving average coefficient. In addition,ε_t is the total returns innovation with conditional variance h_t, given the information Φ_{t-1}; p is the order of the GARCH terms; q is the order of the ARCH term; α_i is the i^{th}-order ARCH coefficient; and β_j is the j^{th}-order GARCH coefficient.

To obtain the housing price dynamic under a risk-neutral measure, we aslo employ an equivalent martingale measure using the conditional Esscher transform developed by Bühlmann et al. (1996). Due to the discount housing price under the Q measure is a martingale, we have:

$$H_{t-1} = E^{Q}\left(\frac{B_{t-1}}{B_t}H_t \middle| \Phi_{t-1}\right) = E^{Q}(\exp(-r)H_t|\Phi_{t-1}) \tag{B.2}$$

We assume that the interest rate is fixed at r. Consequently, $B_t = B_{t-1}e^r$. we obtain:

$$\begin{aligned} H_{t-1} &= e^{-r}E^{Q}(H_t|\Phi_{t-1}) = e^{-r}E^{P}\left(\frac{\Lambda_t}{\Lambda_{t-1}}H_t \middle| \Phi_{t-1}\right) \\ &= H_{t-1}e^{-r}\frac{E^{P}(\exp((a_t+\iota)Y_t)|\Phi_{t-1})}{E^{P}(\exp((a_t)Y_t)|\Phi_{t-1})} \end{aligned} \tag{B.3}$$

Or equivalently,

$$e^r = \frac{E^{P}(\exp((a_t+\iota)Y_t)|\Phi_{t-1})}{E^{P}(\exp((a_t)Y_t)|\Phi_{t-1})} \tag{B.4}$$

In order for risk neutral Q to be an equivalent martingale measure, we need have

$$E^{Q}[\exp(Y_t)|\Phi_{t-1}] = e^r \tag{B.5}$$

Because, Maheu and McCurdy (2004) has point out the conditional moments of return are

$$\begin{aligned} E[Y_t|\Phi_{t-1}] &= u_t \\ Var[Y_t|\Phi_{t-1}] &= h_t + \left(\phi^2+\theta^2\right)\lambda_t = h_t^* \end{aligned} \tag{B.6}$$

Thus, Y_t is normally distributed with mean u_t and variance h_t^*, given the information Φ_{t-1}, we obtain

$$E^{Q}[\exp(\iota Y_t)|\Phi_{t-1}] = \frac{\exp\left((a_t+\iota)u_t + \frac{1}{2}(a_t+\iota)^2 h_t^*\right)}{\exp\left(a_t u_t + \frac{1}{2}a_t^2 h_t^*\right)} = \exp\left((u_t + a_t h_t^*)\iota + \frac{1}{2}h_t^*\iota^2\right) \tag{B.7}$$

Therefore,

$$E^{Q}[\exp(Y_t)|\Phi_{t-1}] = \exp\left(u_t + a_t h_t^* + \frac{1}{2}h_t^*\right) \tag{B.8}$$

Through the Eq. (B.5) and (B.8), we have

$$u_t = r - a_t h_t^* - \frac{1}{2}h_t^* \tag{B.9}$$

Similarly, the characteristic function of ε_t under martingale measure Q is of the form:

$$
\begin{aligned}
E^Q(\exp(i\varpi\varepsilon_t)|\Phi_{t-1}) &= E^P\left(\frac{\Lambda_t}{\Lambda_{t-1}}e^{i\varpi\varepsilon_t}|\Phi_{t-1}\right) = \frac{E^P(e^{a_tY_t}e^{i\varpi\varepsilon_t}|\Phi_{t-1})}{E^P(\exp((a_t)Y_t)|\Phi_{t-1})} \\
&= \frac{\exp(a_tu_t)E^P\left(e^{(a_t+i\varpi)\varepsilon_t}|\Phi_{t-1}\right)}{\exp\left(a_tu_t+\frac{1}{2}a_t{}^2h_t^*\right)} = \frac{\exp\left(\frac{1}{2}(a_t+i\varpi)^2h_t^*\right)}{\exp\left(\frac{1}{2}a_t{}^2h_t^*\right)} \\
&= \exp\left(i\varpi a_th_t^* - \frac{1}{2}\varpi^2h_t^*\right)
\end{aligned}
\tag{B.10}
$$

Consequently, ε_t under the measure Q become normally distributed, with mean $a_th_t^*$ and variance h_t^*, given the information Φ_{t-1}. That is, given the information Φ_{t-1}, $\varepsilon_t^Q = \varepsilon_t - a_th_t^*$ follow normally mean 0 and variance h_t^* under measure Q. Finally, the Eq. (B.1) can be rewritten as:

$$
Y_t = \ln\left(\frac{H_t}{H_{t-1}}\right) = \mu_t + \varepsilon_t = r - a_th_t^* - \frac{1}{2}h_t^* + \varepsilon_t^Q + a_th_t^* \quad = r - \frac{1}{2}h_t^* + \varepsilon_t^Q \tag{B.11}
$$

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